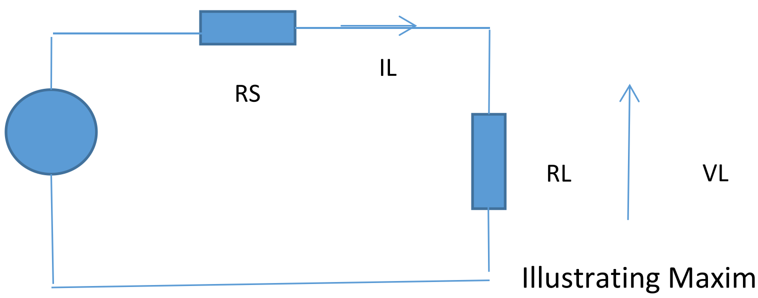
The Maximum Power Transfer Theorem

This is one of the most important theorems in Electrical Engineering. It is particularly important in RF systems where it is necessary to "match" a source to a load to prevent reflections and poor VSWR (Voltage Standing Wave Ratio).

Note that at low frequencies we tend to mismatch in order to provide maximum voltage or current transfer. For the former, we drive a low source impedance into a high load impedance and for the latter, a high source impedance into a low load impedance.

However for maximum power transfer, we require that the source and load impedance are matched: "For maximum power transfer, the source impedance must be the complex conjugate of the load impedance" - a formal definition.

What is meant by the complex conjugate? This means that the reactance of source and load must be opposite. That is, for an inductive source, the load must be capacitive, or *vice versa*. The reactance are thus equal and opposite phasers and therefore cancel. Hence the theorem simplifies; the source and load resistances must be equal (given that the reactance are equal and opposite).



Illustrating Maximum Power Transfer

PROOF

$VL = \frac{RL}{RL + RS}.E$ [potential divider] $IL = \frac{E}{RS + RL}$ [ohms law]

$PL = VL.IL$ [power definition, zero phase shift]

Hence $PL(RL) = E^2.\frac{RL}{RL^2 + RS^2 + 2.RS.RL}$ [substitute for VL and IL]

From differential calculus, maxima of a function:

$PL(RL)\big|_{\max} PL(RL)' = 0$ ... the derivative of the function is zero. Note RL is the dependent variable, RS is fixed.

The expression for PL is a quotient :

$$PL(RL) = \frac{g(RL)}{h(RL)}$$

Hence from the derivative of a quotient, rule :

$$PL(RL)' = \frac{g(RL).h(RL)' - g(RL)'h(RL)}{h(RL)^2}$$

$g(RL) = E^2 RL$ therefore $g(RL)' = E^2$

$h(RL) = RL^2 + RS^2 + 2.RL.RS$ therefore $h(RL)' = 2.RL + 2.RS$

Hence $PL(RL)' = \frac{E^2.RL(2.RL + 2.RS) - E^2(RL^2 + RS^2 + 2.RL.RS)}{(RL^2 + RS^2 + 2.RL.RS)^2} = 0$

[Quotient rule, and maxima]

Separating terms

$$\frac{E^2 RL(2.RL + 2.RS)}{(RL^2 + RS^2 + 2.RL.RS)^2} = \frac{E^2(RL^2 + RS^2 + 2.RL.RS)}{(RL^2 + RS^2 + 2.RL.RS)^2}$$

Cancel 'E' terms and cross multiply denominator on LHS, to obtain

$$2.RL^2 + 2.RL.RS = RL^2 + RS^2 + 2.RL.RS$$

Collecting and cancelling terms

$RL^2 = RS^2$ … and square roots of both sides gives

$RL = RS$ QED

Its a simple but elegant proof and illustrates that sometimes maths is quite useful even for engineers.