Cube problem using node analysis

Consider a cube constructed out of resistors and soldered at the corners. What is the resistance between oposite diagonal corners with a) 1 ohm resistors b) any value resistor. ??

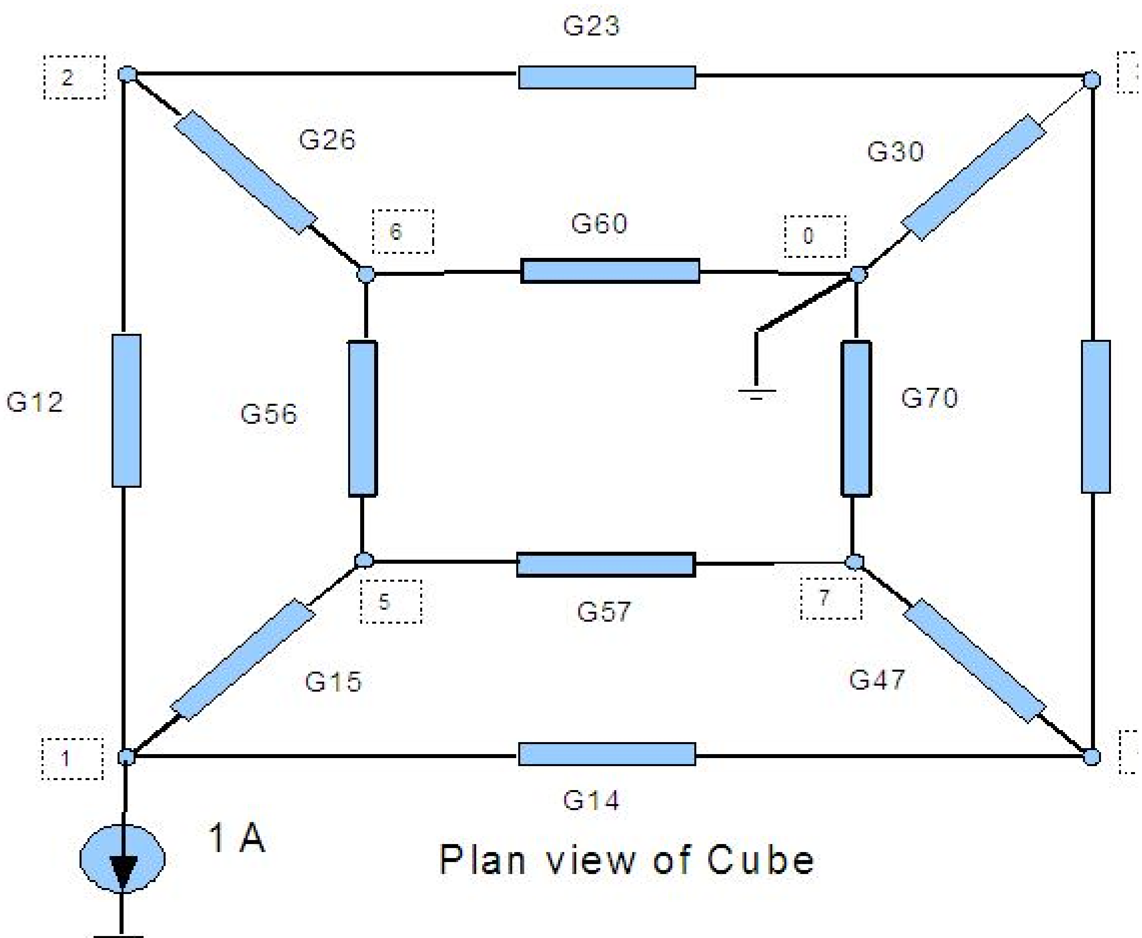
I was made aware of this problem several years ago and initially tried a solution based on repeated application of Rosens Theorem to produce a single equivalent resistor, however I could not get a satisfactory solution - I'm still not sure why this doesn't work.

The standard problem is with one ohm resistors. This easily simplifies taking into account that the network has symetry and the current will split evenly at each corner, and since the resistance is equal, then the potential at the other end of the resistances is the same and can therefore be connected together.

The network therefore simplifies to a seres parrallel conbination of resistances which can easily be calculated, and the answer is 5/6 ohm or 0.866 ohms.

The general problem is considerably harder, ie any value of resistor, but can be solved using node analysis and injecting a current of say 1 amp at the corner of the network (with respect to ground) and connecting the opposite corner to ground. The potential at the current source thus allows the resistance to be easily calculated, ie its the potential divided by 1 amp.

A plan view of the cube network is shown below.



Let the admitances of each of the 12, 1 ohm resistors be as follows :

$G12 := 1 \quad G56 := 1 \quad G15 := 1 \quad G23 := 1 \quad G60 := 1 \quad G26 := 1 \quad G34 := 1 \quad G70 := 1 \quad G30 := 1 \quad G14 := 1 \quad G57 := 1 \quad G47 := 1$

The admittance matrix of the network is therefore :

$$Y1 := \begin{pmatrix} G12+G15+G14 & -G12 & 0 & -G14 & -G15 & 0 & 0 \\ -G12 & G12+G26+G23 & -G23 & 0 & 0 & -G26 & 0 \\ 0 & -G23 & G23+G30+G34 & -G34 & 0 & 0 & 0 \\ -G14 & 0 & -G34 & G14+G47+G34 & 0 & 0 & -G47 \\ -G15 & 0 & 0 & 0 & G15+G56+G57 & -G56 & -G57 \\ 0 & -G26 & 0 & 0 & -G56 & G26+G56+G60 & 0 \\ 0 & 0 & 0 & -G47 & -G57 & 0 & G57+G47+G70 \end{pmatrix}$$

The current vector for the network is just 1 amp at node 1 flowing to ground therefore -ve :

$$I := \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The voltages at each node can thus be calculated :

$$Vn := Y1^{-1} \cdot I$$

$$Y1^{-1} = \begin{pmatrix} 0.833 & 0.5 & 0.333 & 0.5 & 0.5 & 0.333 & 0.333 \\ 0.5 & 0.75 & 0.375 & 0.375 & 0.375 & 0.375 & 0.25 \\ 0.333 & 0.375 & 0.583 & 0.375 & 0.25 & 0.208 & 0.208 \\ 0.5 & 0.375 & 0.375 & 0.75 & 0.375 & 0.25 & 0.375 \\ 0.5 & 0.375 & 0.25 & 0.375 & 0.75 & 0.375 & 0.375 \\ 0.333 & 0.375 & 0.208 & 0.25 & 0.375 & 0.583 & 0.208 \\ 0.333 & 0.25 & 0.208 & 0.375 & 0.375 & 0.208 & 0.583 \end{pmatrix}$$

The resistance is therefore the voltage at node 1 divided by 1 amp:

$$RD := \frac{Vn_0}{I_0}$$

and for equal 1 ohm resistor we get ....

$$RD = 0.833$$

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What about if we let the resitors be 1 through 12 ohms, what now ?

$$G12 := \frac{1}{1}$$

$$G23 := \frac{1}{2}$$

$$G34 := \frac{1}{3}$$

$$G26 := \frac{1}{10}$$

$$G14 := \frac{1}{4}$$

$$G56 := \frac{1}{5}$$

$$G60 := \frac{1}{6}$$

$$G30 := \frac{1}{11}$$

$$G70 := \frac{1}{7}$$

$$G57 := \frac{1}{8}$$

$$G15 := \frac{1}{9}$$

$$G47 := \frac{1}{12}$$

The admittance matrix of the network is as before :

$$Y2 := \begin{pmatrix} G12+G15+G14 & -G12 & 0 & -G14 & -G15 & 0 & 0 \\ -G12 & G12+G26+G23 & -G23 & 0 & 0 & -G26 & 0 \\ 0 & -G23 & G23+G30+G34 & -G34 & 0 & 0 & 0 \\ -G14 & 0 & -G34 & G14+G47+G34 & 0 & 0 & -G47 \\ -G15 & 0 & 0 & 0 & G15+G56+G57 & -G56 & -G57 \\ 0 & -G26 & 0 & 0 & -G56 & G26+G56+G60 & 0 \\ 0 & 0 & 0 & -G47 & -G57 & 0 & G57+G47+G70 \end{pmatrix}$$

The voltages at each node can thus be calculated :

$$Vn := Y2^{-1} \cdot I$$

$$Y2^{-1} = \begin{pmatrix} 5.01 & 4.485 & 3.895 & 4.074 & 2.843 & 2.179 & 1.979 \\ 4.485 & 4.823 & 4.027 & 3.931 & 2.686 & 2.185 & 1.889 \\ 3.895 & 4.027 & 4.72 & 4.046 & 2.371 & 1.879 & 1.804 \\ 4.074 & 3.931 & 4.046 & 5.322 & 2.546 & 1.934 & 2.169 \\ 2.843 & 2.686 & 2.371 & 2.546 & 4.925 & 2.686 & 2.357 \\ 2.179 & 2.185 & 1.879 & 1.934 & 2.686 & 3.762 & 1.415 \\ 1.979 & 1.889 & 1.804 & 2.169 & 2.357 & 1.415 & 4.201 \end{pmatrix}$$

$RD := \frac{Vn_0}{I_0}$

The resistance is therefore the voltage at node 1 divided by 1 amp: and for resistors 1 through 12 ohm resistor we get ....

$$RD = 5.01$$

What about if we let the resitors be 1 through 12 ohms but in a different order, what now ?

$G12 := \frac{1}{1}$

$G23 := \frac{1}{2}$

$G34 := \frac{1}{3}$

$G26 := \frac{1}{4}$

$G14 := \frac{1}{5}$

$G56 := \frac{1}{6}$

$G60 := \frac{1}{7}$

$G30 := \frac{1}{8}$

$G70 := \frac{1}{9}$

$G57 := \frac{1}{10}$

$G15 := \frac{1}{11}$

$G47 := \frac{1}{12}$

The admittance matrix of the network as before :

$$
Y3 := \begin{pmatrix}
G12 + G15 + G14 & -G12 & 0 & -G14 & -G15 & 0 & 0 \\
-G12 & G12 + G26 + G23 & -G23 & 0 & 0 & -G26 & 0 \\
0 & -G23 & G23 + G30 + G34 & -G34 & 0 & 0 & 0 \\
-G14 & 0 & -G34 & G14 + G47 + G34 & 0 & 0 & -G47 \\
-G15 & 0 & 0 & 0 & G15 + G56 + G57 & -G56 & -G57 \\
0 & -G26 & 0 & 0 & -G56 & G26 + G56 + G60 & 0 \\
0 & 0 & 0 & -G47 & -G57 & 0 & G57 + G47 + G70
\end{pmatrix}
$$

The voltages at each node can thus be calculated :

$$Vn := Y3^{-1} \cdot I$$

$$Y3^{-1} = \begin{pmatrix} 4.541 & 3.897 & 3.25 & 3.498 & 2.928 & 2.613 & 1.984 \\ 3.897 & 4.121 & 3.297 & 3.299 & 2.754 & 2.661 & 1.869 \\ 3.25 & 3.297 & 3.957 & 3.431 & 2.329 & 2.167 & 1.762 \\ 3.498 & 3.299 & 3.431 & 4.916 & 2.565 & 2.238 & 2.263 \\ 2.928 & 2.754 & 2.329 & 2.565 & 5.635 & 2.909 & 2.64 \\ 2.613 & 2.661 & 2.167 & 2.238 & 2.909 & 3.843 & 1.621 \\ 1.984 & 1.869 & 1.762 & 2.263 & 2.64 & 1.621 & 4.933 \end{pmatrix}$$

The resistance is therefore the voltage at node 1 divided by 1 amp:

$$RD := \frac{Vn_0}{I_0}$$

$$RD = 4.541$$

Of course to invert a 7 x 7 matrix is extremely long and laborious by hand - a perfect application for software. The work was done by MathCad in this article - one of my favourite tools.

The node analysis method shows how the problem can be tackled and is taught on all undergraduate courses in Electrical Engineering. Node analysis forms the basis of most computer simulation packages, pspice etc. So thats how it is done!

The admittance matraces are shown below.

The amittance Matraces ...

$$Y1 = \begin{pmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 3 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 3 \end{pmatrix}$$

$$Y2 = \begin{pmatrix} 1.361 & -1 & 0 & -0.25 & -0.111 & 0 & 0 \\ -1 & 1.6 & -0.5 & 0 & 0 & -0.1 & 0 \\ 0 & -0.5 & 0.924 & -0.333 & 0 & 0 & 0 \\ -0.25 & 0 & -0.333 & 0.667 & 0 & 0 & -0.083 \\ -0.111 & 0 & 0 & 0 & 0.436 & -0.2 & -0.125 \\ 0 & -0.1 & 0 & 0 & -0.2 & 0.467 & 0 \\ 0 & 0 & 0 & -0.083 & -0.125 & 0 & 0.351 \end{pmatrix}$$

$$Y3 = \begin{pmatrix} 1.291 & -1 & 0 & -0.2 & -0.091 & 0 & 0 \\ -1 & 1.75 & -0.5 & 0 & 0 & -0.25 & 0 \\ 0 & -0.5 & 0.958 & -0.333 & 0 & 0 & 0 \\ -0.2 & 0 & -0.333 & 0.617 & 0 & 0 & -0.083 \\ -0.091 & 0 & 0 & 0 & 0.358 & -0.167 & -0.1 \\ 0 & -0.25 & 0 & 0 & -0.167 & 0.56 & 0 \\ 0 & 0 & 0 & -0.083 & -0.1 & 0 & 0.294 \end{pmatrix}$$